1 The LogRician distribution

If we define the variable

$$Lr(\mathbf{x}) = \log\left(M(\mathbf{x})\right) = \log\left(\sqrt{(A_r + n_r(\sigma_n^2))^2 + (A_i + n_i(\sigma_n^2))^2}\right) \tag{1}$$

this variable follows a LogRician distribution with PDF

$$p_L(x) = \frac{e^{2x}}{\sigma_n^2} e^{-\frac{e^{2x}+A^2}{2\sigma_n^2}} I_0\left(\frac{Ae^x}{\sigma_n^2}\right) \tag{2}$$

with $x \in (-\infty, +\infty)$ and $A^2 = A_r^2 + A_i^2$.

The mean is calculated as

$$E\{x\} = \int_{-\infty}^{\infty} x \frac{e^{2x}}{\sigma_n^2} e^{-\frac{e^{2x}+A^2}{2\sigma_n^2}} I_0\left(\frac{Ae^x}{\sigma_n^2}\right) dx$$

with

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2}.$$

Then

$$\begin{aligned}
E\{x\} &= \int_{-\infty}^{\infty} x \frac{e^{2x}}{\sigma_n^2} e^{-\frac{e^{2x}+A^2}{2\sigma_n^2}} I_0\left(\frac{Ae^x}{\sigma_n^2}\right) dx \\
&= \sum_{k=0}^{\infty} \frac{A^{2k} e^{-\frac{A^2}{2\sigma_n^2}}}{(2\sigma^2)^{2k}(k!)^2\sigma^2} \int_{-\infty}^{\infty} x e^{x(2+2k)} e^{-\frac{e^{2x}}{2\sigma^2}} dx \\
&= \sum_{k=0}^{\infty} \frac{A^{2k} e^{-\frac{A^2}{2\sigma_n^2}}}{(2\sigma^2)^{2k}(k!)^2\sigma^2} (2\sigma^2)^{k+1} \frac{k!}{4} \left[\log(2\sigma^2) + \psi(k+1)\right] \\
&= e^{-\frac{A^2}{2\sigma_n^2}} \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{A^2}{2\sigma^2}\right)^k \frac{1}{k!} \left[\log(2\sigma^2) + \psi(k+1)\right] \\
&= \frac{1}{2}\log(2\sigma^2) + \frac{1}{2}\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \frac{1}{2}\log\left(\frac{A^2}{2\sigma_n^2}\right) \\
&= \frac{1}{2}\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \log(A)
\end{aligned}$$

with $\Gamma^{(l)}(a, b)$ the (lower) incomplete Gamma function. The second order moment is

$$\begin{aligned}
E\{x^2\} &= \int_{-\infty}^{\infty} x^2 \frac{e^{2x}}{\sigma_n^2} e^{-\frac{e^{2x}+A^2}{2\sigma_n^2}} I_0\left(\frac{Ae^x}{\sigma_n^2}\right) dx \\
&= \sum_{k=0}^{\infty} \frac{A^{2k} e^{-\frac{A^2}{2\sigma_n^2}}}{(2\sigma^2)^{2k}(k!)^2\sigma^2} \int_{-\infty}^{\infty} x^2 e^{x(2+2k)} e^{-\frac{e^{2x}}{2\sigma^2}} dx \\
&= \sum_{k=0}^{\infty} \frac{A^{2k} e^{-\frac{A^2}{2\sigma_n^2}}}{(2\sigma^2)^{2k}(k!)^2\sigma^2} (2\sigma^2)^{1+k} \frac{\Gamma(1+k)}{8} \left[(\log(2\sigma^2) + \psi(1+k))^2 + \psi^{1)}(1+k)\right] \\
&= \frac{1}{4}\log^2\left(2\sigma_n^2\right) + \frac{1}{2}\log\left(2\sigma_n^2\right)\left[\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \log\left(\frac{A^2}{2\sigma_n^2}\right)\right] + \frac{1}{4}\tilde{\mathrm{N}}_1\left(\frac{A^2}{2\sigma_n^2}\right)
\end{aligned}$$

with

$$\tilde{\mathrm{N}}_k\left(x\right) = e^{-x}\sum_{i=0}^{\infty}\frac{x^i}{\Gamma(i+1)}\left[\left(\psi(i+k)\right)^2 + \psi^{1)}(i+k)\right].$$

and with $\psi(x)$ is the polygamma function and $\psi^{1)}(x)$ is its first derivative. The variance will be

$$\begin{aligned}\sigma_x^2 &= \frac{1}{4}\left(\tilde{\mathrm{N}}_1\left(\frac{A^2}{2\sigma_n^2}\right) + \left[\Gamma^{(l)}\left(0, \frac{A^2}{2\sigma_n^2}\right) + \log\left(\frac{A^2}{2\sigma_n^2}\right)\right]^2\right) \\ &\approx \frac{1}{4}\left(\tilde{\mathrm{N}}_1\left(\frac{A^2}{2\sigma_n^2}\right) - \log^2\left(\frac{A^2}{2\sigma_n^2}\right)\right)\end{aligned}$$

It is easy to see that

$$\frac{1}{\sigma_x^2} \approx 2\times\frac{A^2}{2\sigma_n^2}.$$

The same solution for the variance may be obtained using a Series expansion on eq. (1)

$$\begin{aligned}Lr(\mathbf{x}) &\approx \log(A) + \frac{n_1}{A} + \frac{n_2^2}{2A^2} - \frac{n_1^2}{2A^2} + \ldots \\ &\approx \log(A) + \frac{n_1}{A} \\ \sigma_{L_r}^2 &= \frac{\sigma^2}{A^2}\end{aligned}$$

2 The Log-Non Central Chi distribution

For parallel acquisitions, if we define the variable

$$Lc(\mathbf{x}) = \log\left(M_L(\mathbf{x})\right) = \frac{1}{2}\log\left(\sum_{l=1}^{L}\left((C_{l_r}(\mathbf{x}))^2 + (C_{l_i}(\mathbf{x}))^2\right)\right) \tag{3}$$

this variable follows a Log-Non central Chi distribution with PDF

$$p_{L_c}(x) = \frac{A_L^{1-L}}{\sigma_n^2}e^{x(N+1)}e^{-\frac{e^{2x}+A_L^2}{2\sigma_n^2}}I_{L-1}\left(\frac{A_L e^x}{\sigma_n^2}\right) \tag{4}$$

with $x \in (-\infty, +\infty)$. If

$$I_{N-1}(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{2k+N-1}}{k!\Gamma(N+k)}$$

the mean is calculated

$$\begin{aligned}E\{x\} &= \int_{-\infty}^{\infty} x\frac{A_L^{1-L}}{\sigma_n^2}e^{x(N+1)}e^{-\frac{e^{2x}+A_L^2}{2\sigma_n^2}}I_{L-1}\left(\frac{A_L e^x}{\sigma_n^2}\right)dx \\ &= \sum_{k=0}^{\infty}\frac{A_L^{2k}e^{-\frac{A_L^2}{\sigma_n^2}}}{2^{N-1+2k}(\sigma^2)^{N+2k}k!\Gamma(N+k)}\int_{-\infty}^{\infty} xe^{x(N+1)-\frac{e^{2x}}{2\sigma^2}+x(N-1+2k)}dx \\ &= e^{-\frac{A_L^2}{\sigma_n^2}}\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{A_L^2}{\sigma_n^2}\right)^k\frac{1}{k!}\left[\log(2\sigma^2) + \psi^{0)}(k+N)\right] \\ &= \frac{1}{2}\log\left(2\sigma_n^2\right) + \frac{1}{2}\frac{A_L^2}{2L\sigma_n^2}\,{}_2F_2\left(1, 1 : 2, 1+L; -\frac{A_L^2}{2\sigma_n^2}\right) + \frac{1}{2}\psi(L)\end{aligned}$$

The second order moment

$$
\begin{aligned}
E\{x^2\} &= \int_{-\infty}^{\infty} x^2 \frac{A_L^{1-L}}{\sigma_n^2} e^{x(N+1)} e^{-\frac{e^{2x}+A_L^2}{2\sigma_n^2}} I_{L-1}\left(\frac{A_L e^x}{\sigma_n^2}\right) dx \\
&= \sum_{k=0}^{\infty} \frac{A_L^{2k} e^{-\frac{A_L^2}{\sigma_n^2}}}{2^{N-1+2k}(\sigma^2)^{N+2k} k!\Gamma(N+k)} \int_{-\infty}^{\infty} x^2 e^{x(N+1)-\frac{e^{2x}}{2\sigma^2}+x(N-1+2k)} dx \\
&= e^{-\frac{A_L^2}{\sigma_n^2}} \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{A_L^2}{2\sigma_n^2}\right)^k \frac{1}{k!} \left[(\psi^2(k+N) + \log(2\sigma^2))^2 + \psi^{1)}(k+N)\right] \\
&= \frac{1}{4}\left[\log^2\left(2\sigma_n^2\right) + 2\log\left(2\sigma_n^2\right)\left(\psi(L) + \frac{A_L^2}{2L\sigma_n^2} {}_2F_2\left(1,1:2,1+N;-\frac{A_L^2}{2\sigma_n^2}\right)\right) + \tilde{\mathrm{N}}_L\left(\frac{A_L^2}{2\sigma_n^2}\right)\right]
\end{aligned}
$$

and the variance

$$
\begin{aligned}
\sigma_x^2 &= \frac{1}{4}\left[\tilde{\mathrm{N}}_L\left(\frac{A_L^2}{2\sigma_n^2}\right) - 2\log\left(2\sigma_n^2\right)\frac{A_L^2}{2L\sigma_n^2} {}_2F_2\left(1,1:2,1+N;-\frac{A_L^2}{2\sigma_n^2}\right)\right. \\
&\quad \left. - \left(\psi(L) + \frac{A_L^2}{2L\sigma_n^2} {}_2F_2\left(1,1:2,1+N;-\frac{A_L^2}{2\sigma_n^2}\right)\right)^2\right] \\
&\approx \frac{1}{4}\left[\tilde{\mathrm{N}}_L\left(\frac{A_L^2}{2\sigma_n^2}\right) - \log^2\left(\frac{A_L^2}{2\sigma_n^2}\right)\right]
\end{aligned}
$$

The same solution for the variance may be obtained using a Series expansion on eq. (3)

$$
\begin{aligned}
Lc(\mathbf{x}) &\approx \log(A_L) + \frac{A}{A_L^2} n\left(L\sigma_n^2\right) + \ldots \\
\sigma_{L_c}^2 &= \frac{\sigma_n^2}{A_L^2}
\end{aligned}
$$